

# A SHARP LOWER BOUND ON THE POLYGONAL ISOPERIMETRIC DEFICIT

E. INDREI <sup>†</sup>

ABSTRACT. A sharp quantitative polygonal isoperimetric inequality is obtained.

## 1. INTRODUCTION

The polygonal isoperimetric inequality states that if  $n \geq 3$  and  $P$  is an  $n$ -gon with area  $|P|$  and perimeter  $L(P)$ , then the deficit is nonnegative,

$$\delta(P) := L^2(P) - 4n \tan \frac{\pi}{n} |P| \geq 0,$$

and uniquely minimized when  $P$  is convex and regular. A full stability result for this classical inequality has recently been obtained in [IN14] via a novel approach involving a functional minimization problem on a compact manifold and the spectral theory for circulant matrices. The heart of the matter is a quantitative polygonal isoperimetric inequality for convex polygons which states that

$$(1.1) \quad \sigma_s^2(P) + \sigma_r^2(P) \lesssim \delta(P),$$

where  $\sigma_s^2(P)$  is the variance of the side lengths of  $P$  and  $\sigma_r^2(P)$  is the variance of its radii (i.e. the distances between the vertices and their barycenter).

The starting point of the proof is the following inequality [FRS85, pg. 35] which holds for any  $n$ -gon:

$$(1.2) \quad 8n^2 \sin^2 \frac{\pi}{n} \sigma_r^2(P) \leq nS(P) - 4n \tan \frac{\pi}{n} |P|,$$

where  $S(P)$  is the sum of the squares of the side lengths of  $P$ . Since  $n^2 \sigma_s^2(P) = nS(P) - L^2(P)$ , it follows that (1.2) is equivalent to

$$(1.3) \quad 8n^2 \sin^2 \frac{\pi}{n} \sigma_r^2(P) \leq \delta(P) + n^2 \sigma_s^2(P).$$

In order to establish (1.1), it is shown in [IN14] that

$$(1.4) \quad \sigma_s^2(P) \lesssim \delta(P)$$

whenever  $P$  is a convex  $n$ -gon; thereafter, a general stability result is deduced via a version of the Erdős-Nagy theorem which states that a polygon may be convexified in a finite number of “flips” while keeping the perimeter invariant. The method of

---

<sup>†</sup> PIRE Postdoctoral fellow.

proof of (1.2) given in [FRS85] is based on a polygonal Fourier decomposition, whereas the technique in [IN14] is based on a third order Taylor expansion of the deficit (in a suitable sense) and as mentioned above involves circulant matrix theory and an optimization problem on a compact manifold. It is natural to wonder whether one can directly deduce (1.1) via the method in [IN14] without relying on [FRS85]. A positive answer is given in this paper. In fact, a new inequality is established which combined with (1.4) improves (1.1).

Let  $\sigma_a^2(P)$  denote the variance of the central angles of  $P$  (i.e. the angles generated by the vertices and barycenter of the vertices of  $P$ , see §2). Then the following is true.

**Theorem 1.1.** *Let  $n \geq 3$  and  $P$  be a convex  $n$ -gon. There exists  $c_n > 0$  such that*

$$c_n \delta(P) \geq \sigma_r^2(P) + |P| \sigma_a^2(P),$$

*and the exponent on the deficit is sharp.*

This result directly combines with (1.4) and yields:

**Corollary 1.2.** *Let  $n \geq 3$  and  $P$  be a convex  $n$ -gon. There exists  $c_n > 0$  such that*

$$c_n \delta(P) \geq \sigma_s^2(P) + \sigma_r^2(P) + |P| \sigma_a^2(P).$$

*Remark 1.3.* The theorem holds for a more general class of polygons. The only requirement in the proof is that the central angles of  $P$  sum to  $2\pi$ .

*Remark 1.4.* An inequality of the form

$$\sigma_a^2(P) \leq c_n \delta(P)$$

cannot hold in general. One can see this by a simple scaling consideration: let  $P$  be a convex polygon and  $P_\alpha$  be the convex polygon obtained by dilating the radii of  $P$  by  $\alpha > 0$ . Then  $\delta(P_\alpha) = \alpha^2 \delta(P)$ , but  $\sigma_a^2(P_\alpha) = \sigma_a^2(P)$ .

Quantitative polygonal isoperimetric inequalities turn out to be useful tools in geometric problems. For instance (1.1) was recently utilized in [CM14] to improve a result of Hales which showed up in his proof of the honeycomb conjecture [Hal01]. Moreover, [IN14] has also been employed in [CN14] to prove a quantitative version of a Faber-Krahn inequality for the Cheeger constant of  $n$ -gons obtained in [BFar]. Related stability results for the isotropic, anisotropic, and relative isoperimetric inequalities have been obtained in [FMP08, FMP10, FI13], respectively.

## Acknowledgements

The author is pleased to acknowledge support from NSF Grants OISE-0967140 (PIRE), DMS-0405343, and DMS-0635983 administered by the Center for Nonlinear Analysis at Carnegie Mellon University. Moreover, the excellent research environment provided by the Hausdorff Research Institute for Mathematics and the Rheinische Friedrich-Wilhelms-Universität Bonn is kindly acknowledged.

## 2. PRELIMINARIES

Let  $n \geq 3$  and  $P \subset \mathbb{R}^2$  be an  $n$ -gon with vertices  $\{A_1, A_2, \dots, A_n\} \subset \mathbb{R}^2$  and center of mass  $O$  which is taken to be the origin. For  $i \in \{1, 2, \dots, n\}$ , the  $i$ -th side length of  $P$  is  $l_i := A_i A_{i+1}$ , where  $A_i = A_j$  if and only if  $i = j \pmod{n}$ ;  $\{r_i := OA_i\}_{i=1}^n$  is the set of radii. Furthermore,  $x_i$  denotes the angle between  $\overrightarrow{OA_i}$  and  $\overrightarrow{OA_{i+1}}$ .

The circulant matrix method introduced in [IN14] is based on the idea that a large class of polygons can be viewed as points in  $\mathbb{R}^{2n}$  satisfying some constraints. More precisely, consider

$$\mathcal{M} := \left\{ (x; r) \in \mathbb{R}^{2n} : x_i, r_i \geq 0, (2.1), (2.2), (2.3) \text{ hold} \right\},$$

where

$$(2.1) \quad \sum_{i=1}^n x_i = 2\pi,$$

$$(2.2) \quad \sum_{i=1}^n r_i = n.$$

$$(2.3) \quad \begin{cases} \sum_{i=1}^n r_i \cos \left( \sum_{k=1}^{i-1} x_k \right) = 0, \\ \sum_{i=1}^n r_i \sin \left( \sum_{k=1}^{i-1} x_k \right) = 0. \end{cases}$$

Note that  $\mathcal{M}$  is a compact  $2n - 4$  dimensional manifold and each point  $(x; r) \in \mathcal{M}$  represents a polygon centered at the origin with central angles  $x$  and radii  $r$ ; therefore, it is appropriate to name such objects *polygonal manifolds*. Indeed, a point  $O$  is the barycenter if and only if

$$\sum_{i=1}^n \overrightarrow{OA_i} = 0,$$

which is equivalent to saying that the projections of  $\sum_{i=1}^n \overrightarrow{OA_i}$  onto  $\overrightarrow{OA_1}$  and  $\overrightarrow{OA_1}^\perp$  vanish; in other words,  $(x; r)$  satisfies (2.3). Furthermore, (2.1) is satisfied by all convex polygons (also many nonconvex ones) and (2.2) is a convenient technical assumption which derives from scaling considerations. Note that the convex regular  $n$ -gon corresponds to the point  $(x_*; r_*) = (\frac{2\pi}{n}, \dots, \frac{2\pi}{n}; 1, \dots, 1)$ . With this in mind, the variance of the interior angles and radii of  $P$  are represented, respectively, by the quantities

$$\sigma_a^2(P) = \sigma_a^2(x; r) := \frac{1}{n} \sum_{i=1}^n x_i^2 - \frac{1}{n^2} \left( \sum_{i=1}^n x_i \right)^2,$$

$$\sigma_r^2(P) = \sigma_r^2(x; r) := \frac{1}{n} \sum_{i=1}^n r_i^2 - \frac{1}{n^2} \left( \sum_{i=1}^n r_i \right)^2.$$

Moreover, in  $(x; r)$  coordinates, the deficit is given by the formula

$$\delta(P) = \delta(x; r) := \left( \sum_{i=1}^n (r_{i+1}^2 + r_i^2 - 2r_{i+1}r_i \cos x_i)^{1/2} \right)^2 - 2n \tan \frac{\pi}{n} \sum_{i=1}^n r_i r_{i+1} \sin x_i.$$

### 3. PROOF OF THEOREM 1.1

By a simple reduction argument, it suffices to prove the inequality on  $\mathcal{M}$ : let  $P$  be a convex  $n$ -gon and note that it is represented by  $(x; r) \in \mathbb{R}^{2n}$ , where  $x \in \mathbb{R}^n$  denotes its interior angles and  $r \in \mathbb{R}^n$  its radii. Convexity implies (2.1), and (2.3) follows from the definition of barycenter. If  $\sum_{i=1}^n r_i = s \neq n$ , consider (by a slight abuse of notation) the polygon  $P_s = (x; \frac{n}{s}r)$  obtained by scaling the radii of  $P$ . Evidently  $\sigma_a^2(P_s) = \sigma_a^2(P)$ ,  $|P_s| = (n/s)^2|P|$ ,  $\sigma_r^2(P_s) = (n/s)^2\sigma_r^2(P)$ ,  $\delta(P_s) = (n/s)^2\delta(P)$ . Hence if the inequality stated in the theorem holds for  $P_s \in \mathcal{M}$ , then it also holds for  $P$ . Now let

$$\begin{aligned} \phi(x; r) &:= n^2(|P|\sigma_a^2 + \sigma_r^2) \\ &= \frac{1}{2} \left( \sum_{i=1}^n r_i r_{i+1} \sin x_i \right) \left( n \sum_{i=1}^n x_i^2 - \left( \sum_{i=1}^n x_i \right)^2 \right) + n \sum_{i=1}^n r_i^2 - \left( \sum_{i=1}^n r_i \right)^2, \end{aligned}$$

and note that it suffices to show

$$(3.1) \quad \phi(x; r) \leq c \delta(x; r)$$

for all  $(x; r) \in \mathcal{M}$ . The polygonal isoperimetric inequality implies  $\delta(x; r) \geq 0$  for every  $(x; r) \in \mathcal{M}$  with  $\delta(x; r) = 0$  if and only if  $(x; r) = z_* := (x_*, r_*)$ . Since  $\mathcal{M}$  is compact and  $\delta$  is continuous it follows that

$$\inf_{\mathcal{M} \setminus B_\delta(z_*)} \delta > 0,$$

and so (3.1) follows easily on  $\mathcal{M} \setminus B_\delta(z_*)$ . Thus it suffices to prove (3.1) for some neighborhood  $B_\delta$  of the point  $z_*$ . Direct calculations imply (recall that the notation is periodic mod  $n$ )

$$(3.2) \quad D\phi(z_*) := (D_x\phi(z_*), D_r\phi(z_*)) = 0,$$

$$D_{x_k x_l} \phi(z_*) = \begin{cases} n(n-1) \sin \frac{2\pi}{n}, & k = l, \\ -n \sin \frac{2\pi}{n}, & k \neq l, \end{cases}$$

$$D_{r_k r_l} \phi(z_*) = \begin{cases} 2(n-1), & k = l, \\ -2, & k \neq l, \end{cases}$$

and  $D_{r_k x_l} \phi(z_*) = 0$ . Thus by letting  $\Phi := D^2 \phi(z_*)$  it follows that

$$\Phi = \begin{pmatrix} n \sin \frac{2\pi}{n} \mathcal{C} & 0_{n \times n} \\ 0_{n \times n} & 2\mathcal{C} \end{pmatrix},$$

where  $0_{n \times n}$  is the  $n \times n$  zero matrix and

$$\mathcal{C} = \begin{pmatrix} n-1 & -1 & & \cdots & & -1 \\ -1 & n-1 & -1 & & \cdots & \\ & -1 & n-1 & -1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \\ & \vdots & \ddots & -1 & n-1 & -1 \\ -1 & & \cdots & & -1 & n-1 \end{pmatrix}_{n \times n}.$$

Moreover,  $D\delta(z_*)$  is given by

$$\begin{cases} D_{x_k} \delta(z_*) = 2n \tan \frac{\pi}{n}, \\ D_{r_k} \delta(z_*) = 0; \end{cases}$$

hence, (2.1) implies

$$\begin{aligned} \langle D\delta(z_*), (x - x_*; r - r_*) \rangle &= \langle D_x \delta(z_*), x - x_* \rangle + \langle D_r \delta(z_*), r - r_* \rangle \\ (3.3) \quad &= 2n \tan \frac{\pi}{n} \sum_{i=1}^n (x_i - (x_*)_i) = 0. \end{aligned}$$

Since  $\phi(z_*) = \delta(z_*) = 0$ , by utilizing (3.2) and (3.3) and performing a third order Taylor expansion it follows that for  $z$  close enough to  $z_*$ ,

$$(3.4) \quad \left| \phi(z) - \frac{1}{2} \langle D^2 \phi(z_*)(z - z_*), (z - z_*) \rangle \right| \leq C |z - z_*|^3,$$

and

$$(3.5) \quad \left| \delta(z) - \frac{1}{2} \langle D^2 \delta(z_*)(z - z_*), (z - z_*) \rangle \right| \leq C |z - z_*|^3,$$

where  $C > 0$ . In particular, there exists  $\eta = \eta(n)$  such that

$$(3.6) \quad \phi(z) \leq \frac{1}{2} \|\Phi\|_2 |z - z_*|^2 + C |z - z_*|^3$$

for all  $z \in B_\eta(z_*)$ . By the results of [IN14, see (iv)' in §3 ], it follows that

$$\inf_{w \in S_{\mathcal{H}}} \langle D^2 \delta(z_*) w, w \rangle =: \sigma > 0,^1$$

where  $\mathcal{H}$  is the tangent space of  $\mathcal{M}$  at  $z_*$  and  $S_{\mathcal{H}}$  is the unit sphere in  $\mathcal{H}$  with center  $z_*$ . Moreover, by continuity there exists a neighborhood  $U \subset \mathbb{R}^{2n}$  of  $S_{\mathcal{H}}$  such that

$$\langle D^2 \delta(z_*) w, w \rangle \geq \frac{\sigma}{2},$$

for all  $w \in U$ . Note that  $\frac{z-z_*}{|z-z_*|} \in U$  for  $z \in \mathcal{M}$  sufficiently close to  $z_*$ . Hence, there exists  $\mu = \mu(\eta, \sigma) \in (0, \eta]$  such that

$$\langle D^2 \delta(z_*) (z - z_*), (z - z_*) \rangle \geq \frac{\sigma}{2} |z - z_*|^2$$

for  $z \in B_\mu(z_*)$ . In particular, for  $\tilde{\mu} := \min\{\mu, \frac{\sigma}{8C}\}$  and  $z \in B_{\tilde{\mu}}(z_*)$ ,

$$\delta(z) \geq \frac{1}{4} \langle D^2 \delta(z_*) (z - z_*), (z - z_*) \rangle;$$

thus, recalling (3.6),

$$\phi(z) \leq \left( \frac{1}{\sigma} \|\Phi\|_2 + \frac{2C}{\sigma} |z - z_*| \right) \langle D^2 \delta(z_*) (z - z_*), (z - z_*) \rangle \leq c_n \delta(z),$$

where  $c_n := \frac{4}{\sigma} \|\Phi\|_2 + \frac{8C}{\sigma} \tilde{\mu}$ . To achieve the second part of the theorem, it suffices to prove the existence of  $c > 0$  such that

$$(3.7) \quad \langle \Phi(x; r), (x; r) \rangle \geq c |(x; r)|^2,$$

for

$$(x; r) \in \mathcal{Z} := \left\{ (x; r) : \sum_{i=1}^n x_i = 0, \sum_{i=1}^n r_i = 0 \right\}.$$

Indeed, if (3.7) holds, let  $\omega : [0, \infty] \rightarrow [0, \infty]$  be any modulus of continuity (i.e.  $\omega(0+) = 0$ ) such that

$$\phi(z) \leq c_n \omega(\delta(z)).$$

Then for  $z \in \mathcal{M}$  close to  $z_*$ , (3.5) implies

$$\delta(z) \leq c_0 |z - z_*|^2,$$

for some  $c_0 > 0$ . Moreover,  $z - z_* \in \mathcal{Z}$  since  $z \in \mathcal{M}$ , and by combining (3.4) with (3.7) it follows that

$$(3.8) \quad \delta(z) \leq c_0 |z - z_*|^2 \leq c_1 \langle \Phi(z - z_*), (z - z_*) \rangle \leq c_2 \phi(z) \leq \tilde{c} \omega(\delta(z)),$$

---

<sup>1</sup>In fact, something stronger is proved: namely that  $\inf_{w \in S_{\mathcal{H}}} \langle D^2 f(z_*) w, w \rangle =: \sigma > 0$  where  $f$  is an explicit function for which  $D^2 f \leq D^2 \delta$ . This is achieved via the spectral theory for circulant matrices and an analysis involving the tangent space of  $\mathcal{M}$  at  $z_*$  and the identification of a suitable coordinate system in which calculations can be performed efficiently. The barycentric condition (2.3) built into the definition of  $\mathcal{M}$  comes up in this analysis.

for some  $\tilde{c} > 0$  provided  $z$  is close to  $z_*$ ; however, since  $\delta(z) \rightarrow 0$  as  $z \rightarrow z_*$  and  $\delta(z) > 0$  for  $z \neq z_*$ , (3.8) leads to a contradiction if

$$\liminf_{t \rightarrow 0^+} \frac{\omega(t)}{t} = 0.$$

Thus the  $\liminf$  is strictly greater than zero and this implies  $\omega$  is at most linear at zero. To verify (3.7), note first that  $\mathcal{C}$  is a real, symmetric, circulant matrix generated by the vector  $(n-1, -1, \dots, -1)$ . A calculation shows that the eigenvalues of  $\mathcal{C}$ , say  $\lambda_k$ , are given by

$$(3.9) \quad \lambda_0 = 0 \quad \text{and} \quad \lambda_k = n \quad \text{for } k = 1, \dots, n-1.$$

Moreover, let  $v_0 := (1, \dots, 1)$ , and for  $l \in \{1, \dots, \lfloor \frac{n}{2} \rfloor\}$  define

$$\begin{aligned} v_{2l-1} &:= \left( 1, \cos \frac{2\pi l}{n}, \cos \frac{4\pi l}{n}, \dots, \cos \frac{2\pi l(n-1)}{n} \right), \\ v_{2l} &:= \left( 0, \sin \frac{2\pi l}{n}, \sin \frac{4\pi l}{n}, \dots, \sin \frac{2\pi l(n-1)}{n} \right). \end{aligned}$$

One can readily check that  $v_k$  is an eigenvector of  $\mathcal{C}$  corresponding to the eigenvalue  $\lambda_{\lceil \frac{k}{2} \rceil}$ , and that the set  $\{v_0, v_1, \dots, v_{n-1}\}$  forms a real orthogonal basis of  $\mathbb{R}^n$  (see e.g. Proposition 2.1 in [IN14]). For  $k = 1, 2, \dots, n$ , define  $b_k := (v_{k-1}; 0, \dots, 0) \in \mathbb{R}^{2n}$  and  $b_k := (0, \dots, 0; v_{k-n-1}) \in \mathbb{R}^{2n}$  for  $k = n+1, \dots, 2n$ . Since the set  $\{b_k\}_{k=1}^{2n}$  forms a real orthogonal basis of  $\mathbb{R}^{2n}$ , given  $(x; r) \in \mathbb{R}^{2n}$  there exist unique coefficients  $\alpha_k \in \mathbb{R}$  such that

$$(x; r) = \sum_{k=1}^{2n} \alpha_k b_k.$$

Thus, by utilizing (3.9) it follows that

$$\begin{aligned} \langle \Phi(x; r), (x; r) \rangle &= \sum_{k, k'=1}^{2n} \alpha_k \alpha_{k'} \langle \Phi b_k, b_{k'} \rangle \\ &= n \sin \frac{2\pi}{n} \sum_{k=1}^n \alpha_k^2 \lambda_{\lceil \frac{k-1}{2} \rceil} |b_k|^2 + 2 \sum_{k=n+1}^{2n} \alpha_k^2 \lambda_{\lceil \frac{k-n-1}{2} \rceil} |b_k|^2 \\ &= n^2 \sin \frac{2\pi}{n} \sum_{k=2}^n \alpha_k^2 |b_k|^2 + 2n \sum_{k=n+2}^{2n} \alpha_k^2 |b_k|^2. \end{aligned}$$

Furthermore, if  $(x; r) \in \mathcal{Z}$ ,

$$\alpha_1 = \frac{\langle (x; r), b_1 \rangle}{|b_1|^2} = \sum_{i=1}^n x_i = 0,$$

$$\alpha_{n+1} = \frac{\langle (x; r), b_{n+1} \rangle}{|b_1|^2} = \sum_{i=1}^n r_i = 0;$$

hence,

$$\begin{aligned} \langle \Phi(x; r), (x; r) \rangle &= n^2 \sin \frac{2\pi}{n} \sum_{k=1}^n \alpha_k^2 |b_k|^2 + 2n \sum_{k=n+1}^{2n} \alpha_k^2 |b_k|^2 \\ &\geq 2n \sum_{k=1}^{2n} \alpha_k^2 |b_k|^2, \end{aligned}$$

and this concludes the proof.

## REFERENCES

- [BFar] M. Bucur and I. Fragala. A Faber-Krahn inequality for the Cheeger constant of N-gons. *J. Geom. Anal.*, to appear.
- [CM14] M. Caroccia and F. Maggi. A sharp quantitative version of Hales' isoperimetric honeycomb theorem. *arXiv:1410.6128* (2014).
- [CN14] M. Caroccia and R. Neumayer. A note on the stability of the Cheeger constant of N-gons. *arXiv:1412.0720* (2014).
- [FI13] A. Figalli and E. Indrei. A sharp stability result for the relative isoperimetric inequality inside convex cones. *J. Geom. Anal.*, 23(2):938–969, 2013.
- [FMP08] N. Fusco, F. Maggi, and A. Pratelli. The sharp quantitative isoperimetric inequality. *Ann. of Math. (2)*, 168(3):941–980, 2008.
- [FMP10] A. Figalli, F. Maggi, and A. Pratelli. A mass transportation approach to quantitative isoperimetric inequalities. *Invent. Math.*, 182(1):167–211, 2010.
- [FRS85] J. Chris Fisher, D. Ruoff, and J. Shilleto. Perpendicular polygons. *Amer. Math. Monthly*, 92(1):23–37, 1985.
- [Hal01] T. C. Hales. The honeycomb conjecture. *Discrete Comput. Geom.*, 25(1):1–22, 2001.
- [IN14] E. Indrei and L. Nurbekyan. On the stability of the polygonal isoperimetric inequality. *arXiv:1402.4460* (2014).

EMANUEL INDREI

CENTER FOR NONLINEAR ANALYSIS  
 CARNEGIE MELLON UNIVERSITY  
 PITTSBURGH, PA 15213, USA  
 EMAIL: [egi@cmu.edu](mailto:egi@cmu.edu)